Beta and Gamma Function

Beta function.

The first Eulerian integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ where $m>0$, $n>0$ is called a Beta Function and is denoted by $\beta(m,n)$

The quantities m, n are positive but not necessarily integers.

Example

i) $\int_0^1 x^3(1-x)^5\,dx$ is Beta fun. and is denoted by $\beta(4,6)$.

ii) $\int_0^1 x^4(1-x)^{\frac{7}{3}}\,dx = \beta\left(5, \frac{10}{3}\right)$

iii) $\int_0^1 x^{\frac{4}{3}}(1-x)^6\,dx = \beta\left(\frac{7}{3}, 7\right)$

iv) $\int_0^1 x^{\frac{2}{5}}(1-x)^{-\frac{1}{2}}\,dx = \beta\left(\frac{7}{5}, \frac{1}{2}\right)$

(v) $\int_0^1 x^{-\frac{2}{3}}(1-x)^{\frac{1}{2}}\,dx = \beta\left(\frac{1}{3}, \frac{3}{2}\right)$

vi) $\int_0^1 x^{\frac{1}{3}}(1-x)^{-2}\,dx$ is not a Beta fun.

$\because m = \frac{4}{3} > 0$ But $n = -1 \not> 0$

* Properties of Beta function.

Property 1: Symmetry of Beta function.

Prove that $\beta(m,n) = \beta(n,m)$

Proof: → $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, $m>0$, $n>0$.

Now $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$\Rightarrow \int_0^1 (1-x)^{m-1}\left(1-(1-x)\right)^{n-1}dx$$

$$= \int_0^1 (1-x)^{m-1}x^{n-1}\,dx = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx \quad n>0,\ m>0$$

$$= \beta(n,m)$$

$$\Rightarrow \beta(m,n) = \beta(n,m)$$

Hence proved.

②

Property 2. Prove that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$$\boxed{= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \,;\, m,n>0}$$

Proof :- $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \,;\, m>0,\ n>0$

Put $x = \frac{t}{1+t}$,

$$dx = \frac{(1+t)(1) - t(1)}{(1+t)^2}\,dt \Rightarrow dx = \frac{1}{(1+t)^2}\,dt$$

Now $x(1+t) = t$

$x + xt = t$

$x = t - xt$

$x = t(1-x)$

$\boxed{t = \frac{x}{1-x}}$

as $x \to 0$

$t \to 0$

as $x \to 1$

$t \to \infty$

$$\Rightarrow \int_0^\infty \left(\frac{t}{1+t}\right)^{m-1}\left(1-\frac{t}{1+t}\right)^{n-1}\left(\frac{1}{(1+t)^2}\right)dt$$

$$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m-1}}\cdot\frac{1}{(1+t)^{n-1}}\cdot\frac{1}{(1+t)^2}\,dt$$

$$\Rightarrow \int_0^\infty \frac{t^{m-1}}{(1+t)^{m-1+n-1+2}}\,dt = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt$$

$$\therefore \quad \beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \quad \left[\because \text{ change variable} \right]$$

also from property 1. Beta fun. is Symmetric

$$\Rightarrow \quad \beta(m,n) = \beta(n,m)$$

$$= \quad \beta(n,m) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{n+m}}\,dx$$

Therefore $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

Property 3. Prove that $\beta(m,n) = \int_0^1 \frac{x^{m+1}+x^{n-1}}{(1+x)^{m+n}}\,dx$, $m>0$, $n>0$

Proof: $\rightarrow$ $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m+1}}{(1+x)^{m+n}}\,dx$$

$$I_1 \quad + \quad I_2$$

where $I_2 = \int_1^\infty \frac{x^{m+1}}{(1+x)^{m+n}}\,dx$

Put $x = \frac{1}{t}$ as $x \to 1$ $t \to 1$

as $x \to \infty$ $t \to 0$

$$dx = -\frac{1}{t^2}\,dt$$

$$I_2 = \int_1^0 \frac{\left(\frac{1}{t}\right)^{n-1}}{\left(1+\frac{1}{t}\right)^{m+n}} \left(-\frac{1}{t^2}\right) dt$$

$$= -\int_1^0 \frac{1}{t^{m-1} \cdot \frac{(1+t)^{m+n}}{t^{m+n}} \cdot t^2}\, dt$$

$$= +\int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\, dt \qquad \left[\because \int_a^b f(x)dx = -\int_b^a f(x)dx\right]$$

$$I_2 = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx \qquad [\text{changing the variable}]$$

$$\Rightarrow \beta(m,n) = I_1 + I_2 = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$= \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$$

Hence proved.

Property 4. Prove that $\beta(m,n) = \beta(m,n+1) + \beta(m$

Proof:→ R.H.S

$\beta(m,n+1) + \beta(m+1,n)$

$$\Rightarrow \int_0^1 x^{m-1}(1-x)^n dx + \int_0^1 x^m (1-x)^{n-1} dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\left[(1-x)^1 + x^1\right] dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1} dx = \beta(m,n)$$

$= L.H.S$

Hence proved.

Property 5. Prove that $\frac{\beta(m,n+1)}{n} = \frac{\beta(m+1,n)}{m}$

Proof $\beta(m,n+1) = \int_0^1 x^{m-1}(1-x)^n dx = \int_0^1 \underset{I}{(1-x)^n}\, \underset{II}{x^{m-1}} dx$

$$= \left[(1-x)^n\left(\frac{x^{m-1+1}}{m-1+1}\right)\right]_0^1 - \int_0^1 n(1-x)^{n-1}(-1)\frac{x^{m-1+1}}{m-1+1} dx$$

$$= (0-0) + \frac{n}{m}\int_0^1 x^m (1-x)^{n-1} dx$$

$$\beta(m,n+1) = \frac{n}{m}\beta(m+1,n) \Rightarrow \frac{\beta(m,n+1)}{n} = \frac{\beta(m+1,n)}{m}$$

Property 6. Prove that $\frac{\beta(m,n+1)}{n} = \frac{\beta(m,n)}{m+n} = \frac{\beta(m+1,n)}{m}$

Proof :> we have

$$\frac{\beta(m,n+1)}{n} = \frac{\beta(m+1,n)}{m} = k \text{ (say)}$$

$$\therefore \beta(m,n+1) = nk \quad , \quad \beta(m+1,n) = mk$$

Also $\beta(m,n) = \beta(m,n+1) + \beta(m+1,n) = nk + mk$

$$k = \frac{\beta(m,n)}{m+n}$$

$$\therefore \frac{\beta(m,n+1)}{n} = \frac{\beta(m,n)}{m+n} = \frac{\beta(m+1,n)}{m}$$

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Property 7. Prove that $\beta(m,n) = \frac{\lfloor m-1 \, \lfloor n-1}{\lfloor m+n-1}$; $m,n>0$ and $m,n \in Z$.

Proof :> $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

$$= \left[x^{m-1}\frac{(1-x)^{n-1+1}}{(n-1+1)(-1)}\right]_0^1 - \int_0^1 (m-1)x^{m-2}.(1).\frac{(1-x)^{n-1+1}}{n-1+1(-1)}dx$$

$$= [0-0] + \frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^n dx$$

$$\beta(m,n) = \frac{m-1}{n}\beta(m-1,n+1) \text{ ——— } (1)$$

changing m to $(m-1)$ and n to $(n+1)$ in ①

$$\beta(m-1, n+1) = \frac{m-2}{n+1} \beta(m-2, n+2)$$

from ①

$$\beta(m,n) = \left(\frac{m-1}{n}\right)\left(\frac{m-2}{n+1}\right) \beta(m-2, n+2)$$

also in similar way

$$\beta(m,n) = \frac{(m-1)(m-2)(m-3)}{(n)(n+1)(n+2)} \beta(m-3, n+3)$$

and so on.

$$\beta(m,n) = \frac{(m-1)(m-2)\ldots(1)}{n(n+1)(n+2)\ldots(n+m-2)} \cdot \beta(1, n+m-1) \quad \text{②}$$

Now $\beta(1, n+m-1) = \int_0^1 x^0 (1-x)^{n+m-2} dx$

$$= \left[\frac{(1-x)^{n+m-1}}{(n+m-1)(-1)}\right]_0^1 = \left(0 - \left(\frac{1}{(n+m-1)(-1)}\right)\right)$$

$$= \frac{1}{n+m-1}$$

$\therefore$ from ② $= \beta(m,n) = \frac{\underline{|m-1} \; [(n-1)(n-2)\ldots 1]}{[1.2.\ldots(n-1)] n(n+1)\ldots(n+m-2)(n+m-1)}$

$$\beta(m,n) = \frac{\lfloor(m-1)\lfloor(n-1)}{\lfloor n+m-1} \quad ; \; m,n>0$$

and $m,n \in Z$.

Hence proved.

Property 8.

Prove that $\beta(m,n) = 2\int_0^{\pi/2} \sin^{(2m-1)}\theta \cos^{(2n-1)}\theta \, d\theta$

Proof :- $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \quad ; \; m,n>0$

Put $x = \sin^2\theta \qquad dx = 2\sin\theta\cos\theta \, d\theta$

$x=0 \qquad x=1$

$\theta=0 \qquad \theta=\frac{\pi}{2}$

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta \, d\theta$$

$$= 2\int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1} \sin\theta\cos\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \; \cos^{2n-2}\theta \; \sin\theta\cos\theta \, d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \; \cos^{2n-1} d\theta$$

Hence proved

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☆ Prove that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$

$$= \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Proof Let

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \int_0^{\pi/2} (\sin^2\theta)^{p/2}(\cos^2\theta)^{q/2} d\theta$$

$$= \int_0^{\pi/2} (\sin^2\theta)^{p/2}(1-\sin^2\theta)^{q/2} d\theta$$

$\sin^2\theta = t$ $\therefore 2\sin\theta\cos\theta \, d\theta = dt$

$$d\theta = \frac{1}{2\sin\theta\cos\theta} dt = \frac{1}{2\sin\theta\sqrt{1-\sin^2\theta}} dt$$

$$= \frac{dt}{2\sqrt{t}\sqrt{1-t}}$$

$\theta = 0$ $t = 0$

$\theta = \frac{\pi}{2}$ $t = 1$

$$= \int_0^1 t^{p/2}(1-t)^{q/2} \frac{1}{2(t)^{1/2}(1-t)^{1/2}} dt$$

$$= \frac{1}{2}\int_0^t t^{\frac{p}{2}-\frac{1}{2}}(1-t)^{\frac{q}{2}-\frac{1}{2}} dt$$

$$= \frac{1}{2}\beta\left(\frac{p}{2}+\frac{1}{2}, \frac{q}{2}+\frac{1}{2}\right) = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Hence proved.

Q.1.

$$\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = 2\beta(m,n)$$

Sol^n.

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \beta(m,n) + \beta(m,n) = 2\beta(m,n)$$

Q.2.

$$\int_0^\infty \frac{x^3}{(1+x)^7} = \frac{1}{60}$$

Sol^n

$$\int_0^\infty \frac{x^{4-1}}{(1+x)^{4+3}}\,dx = \beta(4,3)$$

$$= \frac{\lfloor 4-1 \; \lfloor 3-1}{\lfloor 4+3-1} = \frac{\lfloor 3 \; \lfloor 2}{\lfloor 6} = \frac{(3\times2\times1)(2\times1)}{(6\times5\times4\times3\times2\times1)}$$

2

$$= \cancel{\frac{1}{60}} = \frac{1}{60}$$

$$\Rightarrow \boxed{\int_0^\infty \frac{x^3}{(1+x)^7} = \frac{1}{60}}$$

Q.3 $\int_{-1}^{\infty} \frac{x+1}{(x+2)^6}\,dx = \frac{1}{20}$

Also Solved by Putting $x+1=y$ —

$\frac{x+1}{x+2} = t$

as $x = -1$

$t = 0$

$x+1 = t(x+2)$

$x+1 = tx+2t$

as $x \to \infty$

$t = 1$

$x - tx = 2t - 1$

$x(1-t) = 2t - 1$

$x = \frac{2t-1}{1-t}$

$dx = \frac{(1-t)(2)-(2t-1)(-1)}{(1-t)^2} \Rightarrow dx = \frac{1}{(1-t)^2}dt$

$$\int_0^1 \frac{\left(\frac{2t-1}{1-t}\right)+1}{\left(\frac{2t-1}{1-t}+2\right)^6} \cdot \frac{1}{(1-t)^2}dt$$

$$= \int_0^1 \frac{\left(\frac{2t-\cancel{1}+\cancel{1}-t}{1-t}\right)}{\left(\frac{\cancel{2t}-1+2-\cancel{2t}}{1-t}\right)^6} \cdot \frac{1}{(1-t)^2}dt$$

$$= \int_0^1 \frac{t}{(1-t)^1} \cdot \frac{(1-t)^6}{1} \cdot \frac{1}{(1-t)^2}dt = \int_0^1 t(1-t)^3 dt$$

$$= \beta(2,4) = \frac{\lfloor 1 \lfloor 3}{\lfloor 5} = \frac{1}{20}$$

Q.4. $\displaystyle\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{a^n b^m}\beta(m,n)$

$bx = az$. or $x = \frac{a}{b}z$ $\quad dx = \frac{a}{b}dz$

$$\Rightarrow \int_0^\infty \frac{\left(\frac{a}{b}z\right)^{m-1}}{\left(a + \cancel{b}\left(\frac{az}{\cancel{b}}\right)\right)^{m+n}} \cdot \left(\frac{a}{b}dz\right)$$

$$= \int_0^\infty \frac{\left(\frac{a}{b}\right)^{m-1} \cdot \left(\frac{a}{b}\right) z^{m-1}\,dz}{(a)^{m+n}\,(1+z)^{m+n}}$$

$$= \int_0^\infty \frac{a^{m-1+1}}{a^{m+n}\,b^{m-1+1}} \cdot \frac{z^{m-1}}{(1+z)^{m+n}}\,dz$$

$$= \frac{1}{a^n b^m}\int_0^\infty \frac{z^{m-1}}{(1+z)^{m+n}}\,dz = \frac{1}{a^n b^m}\beta(m,n)$$

Hence proved.

Gamma function.

The Second Eulerian integral $\int_0^\infty e^{-x}x^{n-1}dx, n>0$ is called Gamma function and is denoted by $\Gamma(n)$. The quantity n is +ve but not necessarily integer.

Ex

i) $\int_0^\infty x^3 e^{-x} dx = \Gamma(4)$

ii) $\int_0^\infty x^{2/5} e^{-x} dx = \Gamma\left(\frac{7}{5}\right)$

iii) $\int_0^\infty x^{-1/2} e^{-x} dx = \Gamma\left(\frac{1}{2}\right)$

(iv) $\int_0^\infty x^{-2} e^{-x} dx$ is not a Gamma fun.

$\because \quad n = -2+1 = -1 \not> 0$

Recurrence Formulae for Gamma Fun.

a) if $n>1$ is any real no. then $\Gamma(n) = (n-1)\Gamma(n-1)$

b) if $n>1$ is any integer, then $\Gamma(n) = \lfloor(n-1)$

Proof 1.) $\Gamma(n) = \int_0^\infty \underset{I}{x^{n-1}} \underset{II}{e^{-x}} dx = \left. x^{n-1}\frac{e^{-x}}{-1}\right]_0^\infty$

$$- \int_0^\infty (n-1)x^{n-1}\frac{e^{-x}}{-1}dx$$

$$= (0-0) + (n-1)\int_0^\infty \cancel{(n-1)} x^{n-2}e^{-x}dx$$

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

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(b) $\Gamma(n) = (n-1)\,\Gamma(n-1)$

Replacing n by $n-1$, we get

$$\Gamma(n-1) = (n-2)\,\Gamma(n-2)$$

$$\Rightarrow \quad \Gamma(n) = (n-1)(n-2)\,\Gamma(n-2)$$

in this way

$$\Gamma(n) = (n-1)(n-2)\dots(1)\,\Gamma(1)$$

$$\Gamma(1) = \int_0^\infty x^0 e^{-x}\,dx = \left.\frac{e^{-x}}{-1}\right]_0^\infty = 1$$

$$\Gamma(n) = (n-1)(n-2)\dots(1)$$

$$\Gamma(n) = \underline{|n-1}$$

Properties of Gamma fun.

Property 1. $\Gamma(1) = 1$

$$\Gamma(1) = \int_0^\infty e^{-x} x^0\,dx = \int_0^\infty e^{-x}\,dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^\infty = (0+1) = 1$$

Property 2. $\Gamma(n+1) = n\,\Gamma(n)$

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$$\Gamma(n+1) = \int_0^\infty \underset{II}{e^{-x}} \; \underset{I}{x^n} \, dx$$

$$= x^n \frac{e^{-x}}{-1}\Big]_0^\infty - \int_0^\infty n(x^{n-1}) \frac{e^{-x}}{-1} dx$$

$$= (0-0) + n \int_0^\infty x^{n-1} e^{-x} dx$$

$$\Gamma(n+1) = n\Gamma(n)$$

Property 3 $\Gamma(n+1) = \lfloor n$

$\Gamma(n+1) = n\Gamma(n)$

Similar way $\Gamma(n) = (n-1)\Gamma(n-1)$

$\Rightarrow \quad \Gamma(n+1) = n(n-1)\Gamma(n-1)$

$\Rightarrow \quad \Gamma(n+1) = n.(n-1)(n-2) \cdots (1)\Gamma(1)$

$\Gamma(n+1) = n(n+1) \cdots (1) \qquad [\because \Gamma(1) = 1]$

$= \lfloor n$

Q.1. Show that $\Gamma(n) = \int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx$

$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \qquad$ put $x = \log \frac{1}{y}$

$\frac{1}{y} = e^x$

$y = e^{-x}$

$e^{-x} dx = -dy \quad \Leftarrow \quad dy = e^{-x}(-1)dx$

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$x = 0 \quad y = e^{-0} = 1$

$x \to \infty \quad y = 0$

$$\Gamma(n) = -\int_1^0 \left[\log\left(\frac{1}{y}\right)\right]^{n-1} dy = \int_0^1 \left[\log\left(\frac{1}{y}\right)\right]^{n-1} dy.$$

$$\therefore \Gamma(n) = \int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx.$$

Q.2 Show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Solⁿ Put $x^2 = t \quad 2x\,dx = dt \Rightarrow dx = \frac{dt}{2x}$

$= \frac{dt}{2\sqrt{t}}$

$x = 0 \quad t = 0$

$x \to \infty \quad t \to \infty$

$$\frac{1}{2}\int_0^\infty t^{-1/2} e^{-t}\, dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi} = \frac{\sqrt{\pi}}{2}$$

$\left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right].$

Q.3. Show that $\int_0^\infty x^3 e^{-x} dx = 6$

Solⁿ $\int_0^\infty x^3 e^{-x} dx = \Gamma(4) = \lfloor 4-1 = \lfloor 3 = 6$

☆ Relation between Beta and Gamma func

Prove that $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ where $m>0,\ n>0$

Proof :->

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$$

Put $x = tz$ $\therefore dx = t\,dz$ Now $x = 0 \Rightarrow z = 0$
$x \to \infty \Rightarrow z \to \infty$

$$\Gamma(n) = \int_0^\infty (tz)^{n-1} e^{-tz} (t\,dz) = \int_0^\infty t^n z^{n-1} e^{-tz} dz$$

Multiplying both side by $e^{-t} t^{m-1}$, we get

$$e^{-t} t^{m-1} \Gamma(n) = e^{-t} t^{m-1} \int_0^\infty t^n z^{n-1} e^{-tz} dz$$

or $$\Gamma(n) \cdot e^{-t} \cdot t^{m-1} = \int_0^\infty t^{m+n-1} z^{n-1} e^{-t(z+1)} dz$$

Integrating both side w.r.t t between the limits 0 to ∞

$$\Gamma(n) \cdot \int_0^\infty e^{-t} t^{m-1} dt = \int_0^\infty \left[\int_0^\infty t^{m+n-1} z^{n-1} e^{-t(z+1)} dz\right] dt$$

$$\Gamma(n)\Gamma(m) = \int_0^\infty z^{n-1} \left(\int_0^\infty t^{m+n-1} e^{-t(z+1)} dt\right) dz$$

Put $t(z+1) = y$ as $t = 0 \Rightarrow y = 0$
$t \to \infty \Rightarrow y \to \infty$

$$t = \frac{y}{z+1}$$

$$dt = \frac{dy}{z+1}$$

$$\Rightarrow \Gamma(n)\Gamma(m) = \int_0^\infty z^{n-1} \left(\int_0^\infty \frac{y^{m+n-1}}{(z+1)^{m+n-1}} e^{-y} \frac{dy}{(z+1)}\right) dz$$

$$\Gamma(n)\Gamma(m) = \int_0^\infty \frac{z^{n-1}}{(z+1)^{m+n}} \left(\int_0^\infty y^{m+n-1} e^{-y} dy \right) dz$$

$$= \int_0^\infty \frac{z^{n-1}}{(z+1)^{m+n}} \left(\Gamma(m+n) \right) dz$$

$$= \Gamma(m+n) \int_0^\infty \frac{z^{n-1}}{(z+1)^{m+n}} dz$$

$$\Gamma(m)\Gamma(n) = \Gamma(m+n)\, \beta(m,n)$$

$$\Rightarrow \quad \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$ Hence proved.

★ Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof :⇒ $$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \beta(m,n)$$

Put $m = n = \frac{1}{2}$

$$\frac{\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{\Gamma(1)} = \beta\left(\frac{1}{2}, \frac{1}{2}\right) \qquad [\because \Gamma(1) = 1$$

$$\Rightarrow \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \beta\left(\frac{1}{2}, \frac{1}{2}\right) = \int_0^1 x^{-1/2} (1-x)^{-1/2} dx$$

$$= \int_0^1 \frac{1}{\sqrt{x}\sqrt{(1+x)}} dx$$

Put $x = \sin^2\theta$

$x = 0 \quad \theta = 0$

$x = 1 \quad \theta = \frac{\pi}{2} \qquad dx = 2\sin\theta\cos\theta\, d\theta$

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \int_0^{\pi/2} \frac{2 \sin\theta \cos\theta \, d\theta}{\sqrt{\sin^2\theta}\sqrt{1-\sin^2\theta}} = 2\int_0^{\pi/2} \frac{\sin\theta\cos\theta}{\sin\theta\cos\theta} d\theta$$

$$= 2\left[\theta\right]_0^{\pi/2} = 2\left(\frac{\pi}{2} - 0\right) = \pi$$

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi$$

$$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

Hence proved.

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2} + \frac{q+1}{2}\right)}$$

Duplication formula.

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$

Q. Evaluate $\int_0^{\pi/2} \sin^3 x \cos^{\frac52} x\,dx$

Sol^n. $I = \int_0^{\pi/2} \sin^3 x \cos^{\frac52} x\,dx = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{\frac52+1}{2}\right)}{\Gamma\left(\frac{3+1}{2}+\frac{\frac52+1}{2}\right)}$

$= \frac{1}{2} \frac{\Gamma(2)\,\Gamma\left(\frac74\right)}{\Gamma\left(\frac{15}{4}\right)} = \frac{1}{2} \frac{(1)\Gamma(1).\,\Gamma\left(\frac74\right)}{\frac{11}{4}.\frac74.\Gamma\left(\frac74\right)}$

$= \frac{1}{2} \frac{1}{\frac{11}{4}.\frac74} = \frac{8}{77}$ Ans.

H.W Evaluate $\int_0^{\pi/2} (\sin x)^{\frac23} (\cos x)^{-\frac12} dx$

Q. Show that $\int_0^\infty \frac{1}{1+y^4} dy = \frac{\pi\sqrt2}{4}$

Sol^n Let $I = \int_0^\infty \frac{1}{1+y^4} dy$.

Put $y^2 = \tan t$ $\quad 2y\,dy = \sec^2 t\,dt$

$dy = \frac{\sec^2 t\,dt}{2y} = \frac{\sec^2 t\,dt}{2\sqrt{\tan t}}$

$y = 0 \Rightarrow t \to 0$
$y \to \infty \Rightarrow t \to \frac{\pi}{2}$

$I = \int_0^{\pi/2} \frac{1}{1+\tan^2 t} \frac{\sec^2 t}{2\sqrt{\tan t}} dt = \frac{1}{2}\int_0^{\pi/2} \frac{dt}{\sqrt{\tan t}}$

$$\frac{1}{2}\int_0^{\pi/2} \frac{\sqrt{\cos t}}{\sqrt{\sin t}}\,dt = \frac{1}{2}\int_0^{\pi/2} (\sin t)^{-1/2}(\cos t)^{1/2}\,dt$$

$$= \frac{1}{2}\left[\frac{1}{2}\cdot\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{-\frac{1}{2}+1}{2}+\frac{\frac{1}{2}+1}{2}\right)}\right] = \frac{1}{4}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

$\{\because \Gamma(1) = 1$

$$= \frac{1}{4}\,\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

By Duplication formula. we have.

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)$$

Put $m = \frac{1}{4}$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{\pi}}{2^{-1/2}}\,\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{(2)^{-1/2}}\sqrt{\pi} = \sqrt{2}\,\pi$$

$$I = \frac{1}{4}\left(\sqrt{2}\,\pi\right) = \frac{\pi\sqrt{2}}{4}$$

Hence proved.

H.W $\displaystyle\int_0^{\infty} \frac{y^2}{1+y^4}\,dy = \frac{\pi}{2\sqrt{2}}$

Q $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx = \frac{1}{5005}$

$$\int_0^\infty \frac{x^4+x^9}{(1+x)^{15}}\,dx = \int_0^\infty \frac{x^4}{(1+x)^{15}}\,dx + \int_0^\infty \frac{x^9}{(1+x)^{15}}\,dx$$

$$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}}\,dx + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+5}}\,dx$$

$$= \beta(5,10) + \beta(10,5)$$

$$= 2\beta(5,10) = 2\,\frac{\lfloor 4\,\lfloor 9}{\lfloor 14}$$

$$= 2\cdot\frac{4\cdot 3\cdot \cancel{2}\cdot 1\,\cancel{\lfloor 9}}{\underset{7}{\cancel{14}}\cdot 13\cdot \cancel{12}\cdot 11\cdot \underset{5}{\cancel{10}}\,\cancel{\lfloor 9}}$$

$$= \frac{1}{7\cdot 13\cdot 11\cdot 5} = \frac{1}{5005}$$

Hence proved.

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